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DYNAMIC SIMILITUDE IN INTERNAL-COMBUSTION ENGINES

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Various proposals have been made to adopt as high-speed criterion the quantity $n^2 N_e$ or $n\sqrt{N_e}$ (reference 1) (n = rpm, N_e = effective rated hp), the point made being that this value is approximately constant for type series; that is, unaffected by the engine size. Hereinafter, relations possessing this property will be termed "model quantities," in conformity with Kutzbach's "model rotative speed" (reference 2). K. v. Sanden (reference 1, p. 312) had shown that the cited property can be developed on the "specific rotative speed" $n\sqrt{N_e}$ if the mean pressure and the mean piston speed - or at least its product - are presumed constant. This suggests a method in general form on the basis of similarity considerations, which obviates the assumption of constant piston speed. In fact, it will be seen that the piston speed - as, moreover, any other speed, such as bearing velocity - must be independent of the quantity dimensions and must be a representative quantity similar to the high speed and the specific weight per horsepower (reference 1 (v. Sanden), pp. 312 and 315).

THE LAW OF SIMILITUDE IN INTERNAL-COMBUSTION ENGINES

Notation

λ	length (m)	π	pressure (kg/m^{-2})
τ	time (s)	μ	mass (kg m^{-1} s^2)
κ	force (kg)	ϑ	temperature ($^\circ$K)

The proportional numbers of the other derivated quantities (such as speeds or accelerations) are given in parentheses (v).

*"Ähnlichkeitsbetrachtungen bei Brennkraftmaschinen." Ingenieur-Archiv, Bd. IV, 1933, pp. 373-383.

We start from Newton's general law of similitude in the form (reference 3)

$$\kappa = \mu \frac{\lambda}{\tau^2} \tag{1}$$

which holds for the motion processes of the gases in the working chamber as well as for the motion of the engine parts.

By geometrical similitude the masses vary as the third power of the linear dimensions, or

$$\mu = \lambda^3 \tag{2}$$

to the extent that the gases have the same density and the engine parts are made of the same material. To allow for discrepancies in gas densities, in the structural materials, and eventual structural modifications, the correction factor a is introduced:*

$$\mu = a\,\lambda^3 \tag{2a}$$

For identical operating conditions, the pressures in the working chamber are also identical, because of the cycle, hence

$$\pi \equiv \frac{\kappa}{\lambda^2} = 1 \tag{3}$$

In order to have latitude here also for different graphical representation, the correction factor b is added:

$$\pi \equiv \frac{\kappa}{\lambda^2} = b \tag{3a}$$

From (1), (2a), and (3a) the law of similitude then follows at

$$\frac{\lambda}{\tau} \equiv (v) = \sqrt{\frac{b}{a}} \tag{4}$$

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*Strictly speaking, the motion process of the gases should be separate from that of the moving parts, each with different correction factors a; but as the same law of similitude applies to both processes, this division can be foregone, for the law is conditioned through the requirement (3a), which is the same for both processes (pressure in gas = pressure on piston).

Since λ/τ represents the proportional number of a speed, it follows that the speeds in internal-combustion engines are not affected by the size of the engine and can only be varied within a comparatively narrow zone, characterized by the correction factor $\sqrt{\frac{b}{a}}$. With the length scale λ as basis, we find:

For the time (s),

$$\tau = \sqrt{\frac{a}{b}}\,\lambda = \frac{\lambda}{(v)} \qquad (5)$$

the times vary as the lengths;

For the rotative speeds and angular velocities (min^{-1}, s^{-1}),

$$\tau^{-1} = \sqrt{\frac{b}{a}}\,\lambda^{-1} = (v)\,\lambda^{-1} \qquad (6)$$

the rotative speeds and angular velocities vary inversely to the lengths.

For the accelerations ($m\ s^{-2}$),

$$\frac{\lambda}{\tau^2} = \frac{b}{a}\,\lambda^{-1} = (v)^2\,\lambda^{-1} \qquad (7)$$

the accelerations also vary inversely to the lengths, but the potential variations are greater than by the rotative speeds ($(v)^2$ with respect to (v)).

For the forces (kg),

$$\kappa = b\,\lambda^2 \qquad (8)$$

the forces vary as the squares of the lengths;

For the power output ($kg\,m\ s^{-1}$),

$$\frac{\kappa\lambda}{\tau} = b\sqrt{\frac{b}{a}}\,\lambda^2 = b\,(v)\,\lambda^2 \qquad (9)$$

the performances vary also as the squares of the lengths, with an added possible variation of the speeds (v).

For the weight per horsepower ($kg\ hp^{-1}$),

$$\frac{\mu T}{\kappa\lambda} = \sqrt{\frac{a^3}{b^3}}\ \lambda = (v)^{-3}\lambda \qquad (10)$$

the performances vary as the lengths; the potential variations correspond to the third power of $1/(v)$, hence the speeds are of great influence;

For volumetric efficiency (hp m^{-3}),

$$\frac{\kappa\ \lambda}{T\lambda^3} = b\ (v)\ \lambda^{-1} \qquad (11)$$

it varies inversely to the lengths - the higher the speed, the higher the volumetric efficiency;

For the high speed $(n\sqrt{N_e})$,

$$T^{-1}\sqrt{\frac{\kappa\lambda}{T}} = \sqrt[4]{\frac{b^5}{a^3}} = (v)^2\ \sqrt[4]{ab} \qquad (12)$$

it is not dependent on the dimensions of the machine and is principally affected by the speeds (factor $(v)^2$);

For the specific weight per horsepower (kg $hp^{-3/2}$),

$$\frac{\mu\ T^{3/2}}{(\kappa\lambda)^{3/2}} = \sqrt[4]{\frac{a^7}{b^9}} = \frac{1}{(v)^4\ \sqrt[4]{ab}} \qquad (13)$$

this also is independent of the dimensions and is even more affected by the speeds (factor $(v)^4$) than $(n\sqrt{N_e})$.

MODEL QUANTITIES

Now the model quantities are compiled from the characteristic engine data: rotative speed n (rpm), power N_e (hp/cylinder), quantity V_h (stroke volume in liters per cylinder), and weight G (kg/cylinder). To enable direct comparisons between the individual design types - power, volume, and weight are referred to one cylinder. For identification of the model quantities, underscored letters are used: $\underline{n}$ denotes the rotative speed, $\underline{l}$ the power, $\underline{v}$ the volume, and $\underline{g}$ the weight. Thus the model quantity formed of power and weight, reads $\underline{g}_{\underline{l}}$ in weight comparisons, and $\underline{l}_{\underline{g}}$ in horsepower comparisons.

a) Thermal High Speed

The already employed "high speed" $\underline{n}_l = n\sqrt{N_e}$ - that is, the model quantity formed of rotative speed and horsepower - is termed "thermal high speed" to differentiate it from the corresponding relation between rotative speed and volume. It indicates the rotative speed of the similar one-horsepower engine. It is

$$(\underline{n}_l) = (v)^2 \sqrt[4]{ab} \tag{12}$$

The corresponding relation for the horsepower would be:

$$\underline{l}_n = N_e\ n^2 \tag{12a}$$

the "horsepower of the similar engine with the rotative speed 1/min." As stated, this quantity is partially used, although it affords no values appropriate for the representation.

b) Mechanical High Speed

This expresses the relationship between rotative speed and machine size. As model quantity, obviously

$$\underline{n}_v = n\sqrt[3]{V_h} \tag{14}$$

comes in for consideration. It represents the rotative speed of the similar 1-liter engine. It is

$$(\underline{n}_v) = (v)\ \lambda^{-1}\ \sqrt[3]{\lambda^3} = (v) \tag{14a}$$

the mechanical high speed is a pure speed relation.

The corresponding relation for the volume would be

$$\underline{v}_n = V_h\ n^3 \tag{14b}$$

the "stroke volume of the similar engine with rotative speed 1/min; $\underline{v}_n$ is even less clear than the horsepower relation $\underline{l}_n$ (12a).

c) Specific Weight per Horsepower

This is already defined by

$$\underline{g_l} = \frac{G}{N_e^{3/2}} \tag{13a}$$

It represents the weight of the similar one-horsepower engine. According to what has gone before,

$$(\underline{g_l}) = \frac{1}{(v)^4 \sqrt[4]{ab}} \tag{13}$$

The reciprocal relation would be

$$\underline{l_g} = \frac{N_e}{G^{2/3}} \tag{13b}$$

the "horsepower of the similar engine of 1-kilogram weight." Equation (13a) should be more appropriate for representation.

d) Weight per Liter of Stroke Volume

This is the only relation with linear quantities

$$\underline{g_v} = \frac{G}{V_h} \tag{15}$$

It represents the weight of the similar 1-liter engine. The ratio of the liter weights is

$$(\underline{g_v}) = \frac{\mu}{\lambda^3} = a \tag{15a}$$

The reciprocal relation

$$\underline{v_g} = \frac{V_h}{G} \tag{15b}$$

gives "stroke capacity of the similar engine of 1-kilogram weight."

e) Specific Horsepower (hp/cu in.)

It is the model quantity formed of horsepower and stroke capacity and expressed by

$$\underline{l}_{\underline{v}} = \frac{N_e}{V_h^{2/3}} \tag{16}$$

which is the horsepower of the similar 1-liter engine. The ratio of the specific horsepowers is

$$(\underline{l}_{\underline{v}}) = \frac{b(v)\,\lambda^2}{\lambda^2} = b(v) \tag{16a}$$

The reciprocal relation may also become valuable:

$$\underline{v}_{\underline{l}} = \frac{V_h}{N_e^{3/2}} \tag{16b}$$

it indicates the "stroke volume of the similar one-horsepower engine."

f) The Relation between Rotative Speed and Weight

This involves

$$\underline{n}_{\underline{g}} = n\sqrt[3]{G} \tag{17}$$

or

$$\underline{g}_{\underline{n}} = G\,n^3 \tag{17a}$$

although no particular importance is likely to be attached to them.

We explore in this connection the commonly used characteristic values.

Model quantities in the sense of the similarity consideration are: the mean pressure (working pressure), the mean piston speed, the weight per stroke volume G/V_h, the piston loading N_e/F_k, and the charge utilization N_e/nV_h. Weight per horsepower G/N_e, and horsepower per stroke volume N_e/V_h are not model quantities; the first increases, the other decreases with the size of the machine.

Incidentally, the nondimensional relation of the specific horsepower $\underline{l}_{\underline{v}} = N_e/V_h^{2/3}$ is employed by Italy against Held (Automot. Ind., No. 16, 1930), and Stadie (Automob.-techn. Z., 1932, p. 356).

K. v. Sanden has already used the quantities n_l and g_l in some of his studies (reference 1, p. 317). The corresponding relations $n_l = n\sqrt{N_e}$ and $g_l = G/N_e^{3/2}$ manifest that the engines in the logarithmic diagram must group around the straight line $n_l^3\, g_l = n^3 G$ = constant since, according to (17a) $n^3 G$ itself is a model quantity.

By plotting the thermal high speed n_l against the mechanical high speed n_v, both in logarithmic scale, a further system of curves - sloped at 45° - can be added, which gives the specific horsepower. This chart (fig. 1) contains the data for various engines.

The chosen method has the advantage of bringing out the mean pressures also. For 4-stroke-cycle engines, it is

$$p_e = \frac{900\ N_e}{V_h n} = 900\ \frac{n^2}{n^3}\frac{N_e}{V_h}$$

or

$$p_e = 900\ \frac{n_l^2}{n_v^3} \tag{18}$$

The lines p_e = constant in the logarithmic chart are therefore straight lines with a slope $\tan \alpha = 3/2$. For 2-stroke-cycle or double-acting engines the p_e lines are shifted upward or, as shown in the chart, the plotted engines are shifted downward by the amount $\ln \sqrt{2} = \frac{1}{2} \ln 2$ for 2-stroke-cycle or double-acting engines, and by $\ln \sqrt{4} = \ln 2$ for 2-stroke-cycle and double-acting engines.

Examples

1. Aircraft engine.- 12-cylinder, 4-stroke-cycle; N_e = 680 hp, n = 1650 rpm, V_h = 46.9 liters. The corresponding values per cylinder are: N_e = 56.7 hp/cylinder, V_h = 3.91 liters/cylinder. Hence, $n_l \equiv n\sqrt{N_e}$ = 12,410; $n_v \equiv n\sqrt[3]{V_h}$ = 2600; $l_v \equiv \frac{N_e}{V_h^{2/3}}$ = 22.70; $p_e \equiv 900\ \frac{n_l^2}{n_v^3}$ = 7.91 kg/cm². The last two values can be read immediately from the chart.

2. Ship engine.- 6-cylinder, 2-stroke-cycle - N_e = 2850 hp, n = 160 rpm, V_h = 1278 liters; each cylinder: N_e = 475 hp, V_h = 213 liters. Hence, n_l = 3490, n_v = 956, l_v = 13.3, $p_e = \frac{900}{2} \frac{n_l^2}{n_v^3} = 6.27 \text{ kg/cm}^2$. The last value is read from the p_e lines, the engine plot being shifted downward by $\frac{1}{2}$ ln 2.

Another chart, featuring the engine weight, as obtained from the weight characteristics g_l and g_v (fig. 2), shows the weight of the similar one-horsepower engine, g_l, plotted against g_v, that is, the weight of the similar 1-liter engine. The specific horsepower is again read from the 45°-slope lines

APPLICATION TO ENCOUNTERED PHENOMENA

a) Gravity Phenomena

Cooling water and oil circulation are influenced by the force of gravity. Froude's model law is applicable. For the ratio of the Froude characteristics $F = v^2/lg$ (g = acceleration due to gravity), we find:

$$(F) \equiv \frac{(v)^2}{(l)(g)} = \frac{b/a}{\lambda\ 1} = \frac{b}{a}\lambda^{-1} = (v)^2\lambda^{-1} \qquad (19)$$

The gravity phenomena do not proceed similarly; the Froude characteristics decrease with increasing engine size.

b) Phenomena Due to Elastic Forces

This includes the stresses in the gears and casing so far as they are attributable to the cycle of action of the engine (hence not stresses due to gravity, but the stresses in connecting rods, crankshafts, bearings, tie rods, cylinder heads, and those due to gas pressure in the cylinder). It includes further all vibration phenomena traceable to elastic forces, such as critical crankshaft rpm, resonance vibrations of the gas columns in the inlet and exhaust lines, etc.

Cauchys' model theorem is applicable: For $C = \frac{v}{\sqrt{E/\rho}}$ (E = modulus of elasticity and shear modulus, respectively, in kg m^{-2}, ρ = density in kg $m^{-4}s^2$), we have:

$$(C) \equiv \frac{(v)}{1} = (v) = \sqrt{\frac{b}{a}} \tag{20}$$

The phenomena due to elastic forces are similar; they are unaffected by the size of the engine. In other words, if the crankshaft of a machine has a critical rpm 15 percent above the rated rpm for full load, the crankshaft of a machine of the same type series but different size, has a critical rpm which is also 15 percent above the rated rpm of this second machine. Or, if the gas column in the exhaust line is in resonance at the rated rpm, a similar larger machine will also manifest resonance at the rated rpm. Vibration phenomena in the fuel lines of solid-injection engines manifest at the rated rpm, the same variation for any engine size when referred to degrees of crank angle.

c) Flow Phenomena under the Influence of Inertia and Viscosity

Hereto belong all flow phenomena accompanying low pressures, hence those in the inlet and exhaust lines and, chiefly, the scavenging processes in 2-stroke-cycle engines. The cooling flows in air-cooled engines can also be included so long as the inflow velocities are the same.

Here the Reynolds model law is applicable: For $R = \frac{vl}{\nu}$ (ν = kinematic viscosity in m^2s^{-1}), it is

$$(R) \equiv \frac{(v)(l)}{1} = \sqrt{\frac{b}{a}}\,\lambda = (v)\lambda \tag{21}$$

The flow phenomena in internal-combustion engines, therefore, do not vary similarly, since the Reynolds number increases with increasing size of the machine. Thus the scavenging curve of a small machine could differ from that of a large machine, in spite of identical scavenging processes and identical arrangements of the scavenging elements. More elaborate investigations are needed on this subject. According to aerodynamic researches (reference 4), the effect of the Reynolds number on the flow variation is not great; but it is possible that the turbulence processes, which are of equal importance for the scavenging (reference 5), are more intimately related to the Reynolds number and hence might manifest appreciable differences. This should be decided by experiments.

d) Thermic and Thermodynamic Phenomena

This involves the temperatures in the working medium and in the surrounding parts (cylinder walls, pistons, and further, in the cooling water), the heat exchange in the gas, the heat transfer in the vicinity, and the heat dissipation. The temperature in the operating medium is related to the pressure through the polytropic equation

$$\frac{T}{p^{\frac{m-1}{m}}} = \text{const} \tag{22}$$

In the comparisons, of course, equal starting points must be assumed (equal initial pressure and temperature) - also permissible in this summary analysis. The polytropic exponent of the internal-combustion engine is slightly less than the adiabatic factor κ (reference 6), not only because of the heat transfer on the walls but also as a result of the variation of the specific heat at higher temperatures. Its absolute value, however, is not of interest here.

For the proportional numbers, it therefore gives:

$$\vartheta = \pi^{\frac{m-1}{m}} = b^{\frac{m-1}{m}} \tag{23}$$

In the comparison of heat-conductivity processes, the Fourier and the Péclet model laws must be allowed for - one for the temperature distribution and its time rate of change in the surrounding walls, the other for the exchange of heat in the working chamber itself.

For $Fo = \frac{l^2}{ta_o}$, where $a_o = \frac{\lambda_W}{c\gamma}$ is the temperature conductivity in $m^2 s^{-1}$ (λ_W = heat conductivity, c = specific heat, γ = unit weight), we have:

$$(Fo) = \frac{\lambda^2}{\tau\ 1} = \sqrt{\frac{b}{a}}\ \lambda = (v)\ \lambda \tag{24}$$

if identical materials - that is, $(a_o) = 1$ - are used.

The heat distribution and its variation with respect to time is therefore not similar in the walls, because the Fourier characteristics increase with increase in size of the machine. For instance, on large machines the variation with respect to time lags behind that on a small one, when referred to the period of the machine - say, of crank angles.

For Péclet's number $P = \frac{vl}{a_o}$, it affords in the same manner

$$(P) = \frac{(v)\lambda}{(a_o)} = \frac{1}{b} (v) \lambda = \frac{1}{a} (v)^{-1} \lambda \qquad (25)$$

In this instance, however, the temperature conductivities $a_o = \frac{\lambda_W}{c_p \gamma}$ are not equal, but linearly dependent upon the pressure (reference 1, p. 499), whence the ratio (a_o) becomes equal to the ratio π of the pressures, that is, equal to b. This value is entered in (25).

The exchange of heat in the working medium is therefore not similar because the Péclet numbers increase with the size of the machine. The exchange in larger machines is slower and so much more, as the speeds, factor $(v)^{-1}$, are lower.

Important also are the proportional numbers for the heat quantities passing to the walls and carried from there in the cooling water. For comparisons, these heat quantities must be considered as parts of the total heat input, hence they are referred to the horsepower N_e of the machine (while tacitly assuming equal efficiencies).

For the quantity of heat $q''_{\ddot{u}}$ transmitted in unit time to the walls (surface F) by contact, we have:

$$q''_{\ddot{u}} = \alpha \, F \, \Delta \, T$$

By turbulent flow, as undoubtedly is the case in the internal-combustion engine, the transition factor α can be expressed in the form (reference 3, p. 498):

$$\alpha = K \, P^{3/4} \, \frac{\lambda_o}{L_o}$$

where K is a constant factor dependent upon the form and type of flow, P the Péclet number, λ_o the heat conductivity, and L_o a linear dimension.

Hence, by equal heat conductivity (i.e., $(\lambda_o) = 1$), the ratio of heat quantities transmitted per unit horsepower, is

$$\frac{(q_u'')}{(N_e)} = \frac{1\ (P)^{3/4}\ 1\ \lambda^2\ (\Delta T)}{\lambda b \sqrt{\frac{b}{a}}\ \lambda^2} = \frac{(\Delta T)}{a^{7/4}\ (v)^{15/4}}\ \lambda^{-1/4} \qquad (26)$$

P being expressed by the value found in equation (25).

The heat transmissions by contact are therefore not similar. At equal temperatures, $(\Delta T) = 1$, the heat transfer in larger machines is somewhat less than in a small one* (factor $\lambda^{-1/4}$). With Nusselt's heat transmission formula (reference 7),

$$\alpha = 0.99 \sqrt[3]{p^2\ T}\ (1 + 1.24\ c_m)$$

(c_m = mean piston speed) affords

$$\frac{(q_u'')}{(N_e)} = \frac{\sqrt[3]{b^2 b^{\frac{n-1}{n}}}\ \lambda^2\ (\Delta T)}{b \sqrt{\frac{b}{a}}\ \lambda^2} = \frac{(\Delta T)}{a^{1/3n} (v)^{1+2/3n}} \qquad (27a)$$

if the first term only is considered, and

$$\frac{(q_u'')}{(N_e)} = \frac{(\Delta T)}{a^{1/3n}\ (v)^{2/3n}} \qquad (27b)$$

so far as the second term only is considered.

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*This may seem surprising at first glance, since the heat-transfer coefficient increases with higher gas velocities. But it should be remembered that we must always figure with equal conditions. If the gas velocities are higher, the other velocities must also be higher and the time intervals correspondingly shorter; hence the heat transfer less. The formula indicates that this influence predominates.

In both cases the quantity of heat transmission per unit horsepower, according to this formula, would be independent of the size of the machine, but related in the same sense, as through equation (26) - only weaker - to the speeds.*

The heat-transfer factor by radiation, is

$$\alpha_s = \frac{\left(\frac{T_2}{100}\right)^4 - \left(\frac{T_1}{100}\right)^4}{T_2 - T_1} C_{12}$$

where

T_2 gas temperature

T_1 wall temperature

C_{12} effective radiation factor

At equal temperatures, the ratio of the heat quantities transmitted per unit horsepower, is

$$\frac{(q_s)}{(N_e)} = \frac{1\ \lambda^2}{b\sqrt{\frac{b}{a}}\,\lambda^2} = \frac{1}{a(v)^3} \qquad (28)$$

The heat transfers by radiation vary similarly; they are unaffected by the size of the machine. At high speeds, relatively less heat is transferred (factor $(v)^{-3}$).

The quantity of heat carried off through the cylinder walls per unit time is

$$q = \frac{\lambda_0\ F\ \Delta\ T}{\delta}$$

where λ_0 is the heat conductivity, δ the wall thickness. Here the ratio $(q)/(N_e)$ (assuming identical materials, hence $(\lambda_0) = 1$) is:

$$\frac{(q)}{(N_e)} = \frac{1\ \lambda^2\ (\Delta T)}{\lambda\ b\sqrt{\frac{b}{a}}\,\lambda^2} = \frac{(\Delta T)}{a(v)^3}\,\lambda^{-1} \qquad (29)$$

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*Experiments might afford information on the scope of validity of the different heat-transfer formulas.

The heat removal is not uniform; large machines remove relatively less heat than small ones (factor λ^{-1}), hence the pistons must be cooled, starting from a certain size. At higher speeds, the heat removal decreases.

Translation by J. Vanier,
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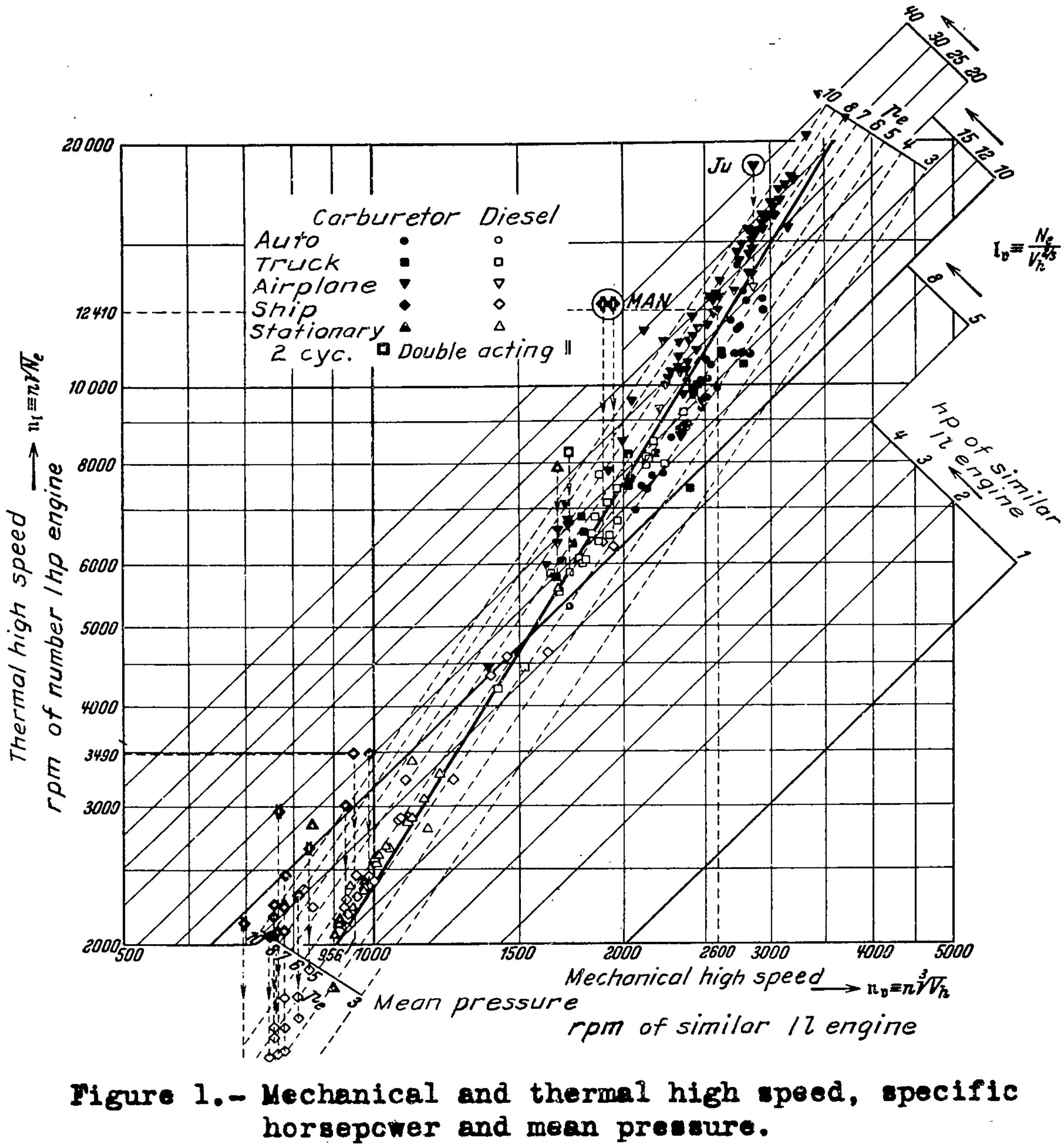

Figure 1.- Mechanical and thermal high speed, specific horsepower and mean pressure.

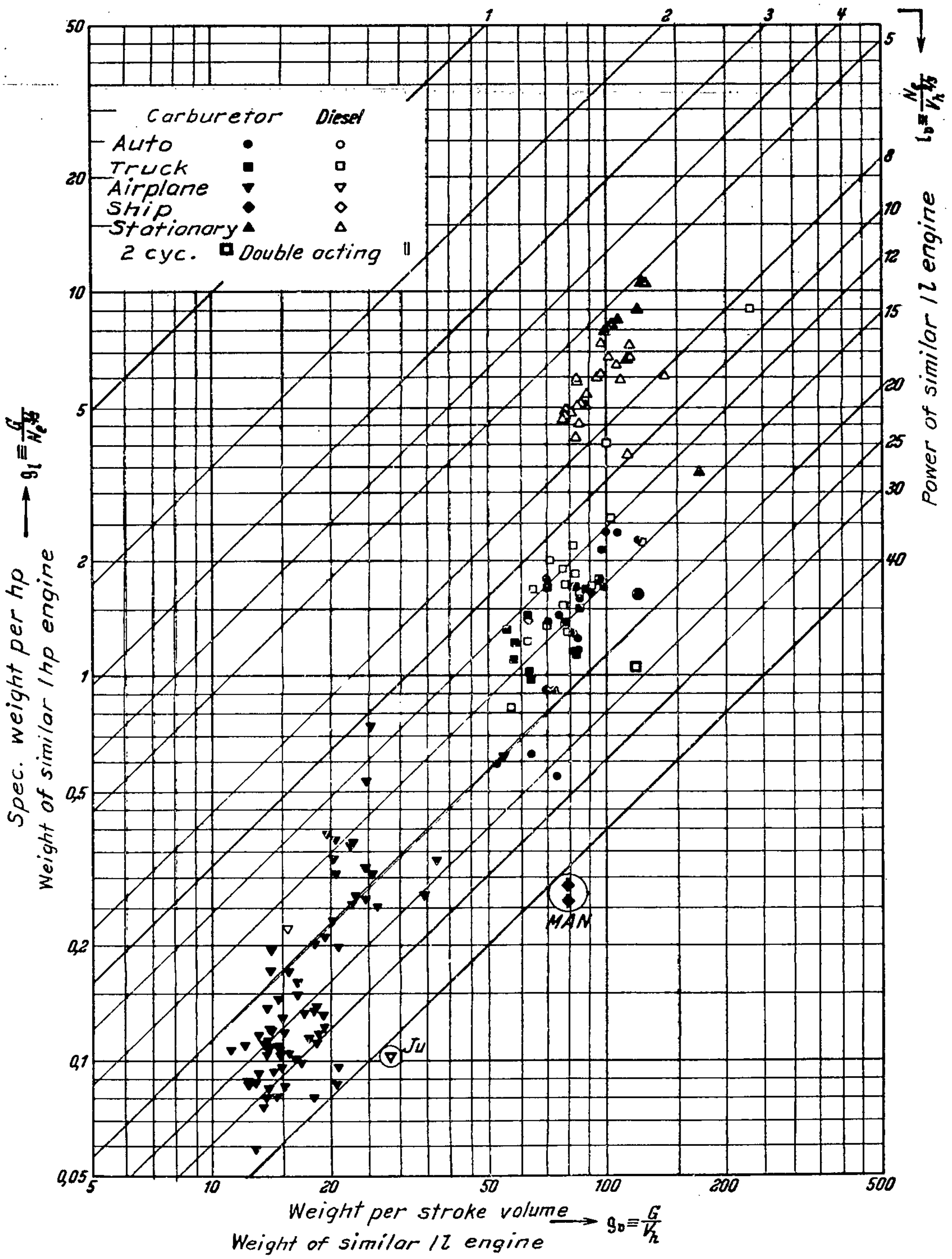

Figure 2.- Weight per stroke capacity, specific weight per horsepower and specific horsepower per stroke capacity.